Outline

1) Vectors vector fields & derivations
2) Lie bracket = Lie derivative
3) Lie algebras
4) Frobenius
5) Covectors
6) Sym^2 + metrics

① Lem 1) $T_0\mathbb{R}^d \longrightarrow \{$Derivations from $C^\infty(\mathbb{R}^d) \to \mathbb{R}_0\}$ is an isomorphism.

$= \{D : C^\infty(\mathbb{R}^d) \to \mathbb{R}$ $\mathbb{R}$-linear s.t.

$D(fg) = f(0)D(g) + g(0)D(f)\}$

⌈ Inverse given by $D \mapsto \sum D(x^i)\frac{\partial}{\partial x^i}\Big|_0$ Check harder direction:

$Df = \sum D(x^i)\frac{\partial f}{\partial x^i}\Big|_0 \quad \forall f$: $\quad f(x) - f(0) = \int_0^1 \frac{d}{dt} f(tx)\, dt$.

Taylor $\Rightarrow$ $f(x) = f(0) + \sum_i x_i g(x_i)$, $g(0) = \frac{\partial f}{\partial x_i}(0)$

∘ $D(1) = D(1^2) = 2D(1) \Rightarrow 0$

$\Rightarrow D(f(0)) = 0$

∘ $Df = \dots$ ⌋

2) If $\varphi : S \to \mathbb{R}^d$, Vect along $\varphi \Rightarrow Der(C^\infty(\mathbb{R}^d), C^\infty(S)_\varphi)$

⌈ $V := D(x^i)\frac{\partial}{\partial x^i}$. Check equality at $p \in S$ $\forall p$

wlog $\varphi(p) = 0$

$f = f(0) + \sum x_i g_i(x)$

$Df = \sum D(x_i)\frac{\partial f}{\partial x^i}$ ⌋

Prop If $\varphi: S \to \check{S}$, Vector fields along $S \longrightarrow \mathrm{Der}(C^\infty(\check{S}), C^\infty(S)_\varphi)$ is an isomorphism. $\ulcorner$ To run above arg, need bump fn.

Prop $\mathrm{Vect}(S) \to \mathrm{Der}(C^\infty(S), C^\infty(S))$ is an isom.

$\ulcorner$

Bump fn $V^i = D(\eta x^i)$

$\eta =$

$\hat{\eta} =$

$$D(\hat{\eta})_p = D(\eta\hat{\eta})_p = D(\eta)_p + D(\hat{\eta})_p$$
$$\Rightarrow D(\eta)_p = 0$$

$\lrcorner$

(2)

Lie bracket Operation on fns $[X,Y]f = XYf - YXf$ Check Der of $C^\infty(S)$!

Computation
- $\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0$
- $[X,Y] = -[Y,X]$
- $[Y, fX] = Yf + f[Y,X]$

Thm $\mathcal{L}_X Y = [X,Y]$ (clearly skew symmetric)

$\ulcorner$ $\Phi: U \times (-\varepsilon, \varepsilon) \to S$ local flow of X @ p.

given f on S, compute $(\mathcal{L}_X Y)f$

Taylor $\Rightarrow$ $f(\Phi(q,t)) = f(q) + t\, g(q,t)$ w/ $g = Xf$

$$(\mathcal{L}_X Y)f\big|_p = -\lim_{t\to 0} \frac{d\Phi_t Y_{\Phi_{-t}p} - Y_p}{t} f$$

$$= -\lim_{t\to 0} \frac{Y_{\Phi_{-t}p}(f\circ\Phi_t) - Y_p f}{t}$$

$$= -\lim_{t\to 0} \frac{Y_{\Phi_{-t}p}(f + tg) - Y_p f}{t}$$

$$= -\lim_{t\to 0} Y_{\Phi_{-t}p}\, g - \lim_{t\to 0} \frac{Y_{\Phi_{-t}p} f - Y_p f}{t}$$

$$= -YXf + XYf$$

(3)

Lie algebras

$\mathcal{L}_X(fY) = f(\mathcal{L}_X Y) + \mathcal{L}_X f\, Y$

Defn V vect space w/ bracket $[\,,\,]$

- bilinear
- anti-comm
- jacobi ident.

Eg 1) $Vect(S)$

2) $Lie(G)$

$\leftarrow$ X, Y left-invt $\Rightarrow$ $[X,Y]$ left-invt)

φ diffeo induces othy isom $\Rightarrow$ Derivations, etc natural

(4)

Frobenius

Defn A distribution is a subbundle $E \subseteq TS$

Defn E is involutive if $X, Y \in E \Rightarrow [X,Y] \in E$

Defn An integral mfld is $M \subseteq S$ s.t. $TM_q \subseteq E_q$ $\forall\, q \in M$.

Defn E rank r is completely integrable if local submersion?

Q Why equivalent $\exists$ integral r-mfld thru each point.

r. $\mathfrak{h} \subseteq \mathfrak{g}$ closed under $[\,,\,]$; the corresp left-invt

Ex If $\cdots = 0$
dist is comp. int.

Thm Completely integrable $\Leftrightarrow$ involutive

[① Case $[E_i, E_j] = 0$

let I^{n-r} be transversal construct local map

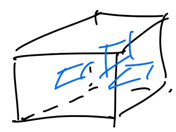
$$I^{n-r} \times \mathbb{R}^r \longrightarrow M$$

+ show it's a diffeo:

(1 at a time)

· invert

② General case



coords $\frac{\partial}{\partial x^1} \cdots \frac{\partial}{\partial x^r}, \frac{\partial}{\partial x^{r+1}} \cdots \frac{\partial}{\partial x^n}$

choose $E_i = \frac{\partial}{\partial x^i} + c_{ij} \frac{\partial}{\partial x^j}$ $\quad i \in (1 - r)$, $j \in (r+1 - n)$

$\Rightarrow [E_i, E_j] \in \text{span}\,(\text{——})$

$\Rightarrow \qquad = 0.$

]

Ex If $\mathfrak{h} \subseteq \mathfrak{g}$ is a sub-algebra, then the comp dist is integrable, and the maximal connected integral field thru e gives a sub gp H

[If $h \in H$,

$L_{h^{-1}}(H) = H$ (by [illegible] [illegible] by maximal

$h_1^{-1} h_2 \in H$ $\Rightarrow$ subgp

$\forall h_1, h_2$

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